

THE ARITHMETIC OF INTEGERS

- multiplication, exponentiation, division, addition, and subtraction

What to do and what *not* to do.

THE INTEGERS

and their operations

Recall that an **integer** is one of the *whole numbers*, which may be either positive, negative, or zero.

In that sense, an integer is simply either a natural number with a positive or negative sign attached to it or the number 0 which may also be considered a signed number.

The integer number system has five arithmetical operations.

We will study all five in detail in this tutorial.

Exponentiation (a special case of multiplication) will also be covered as a separate topic because of its importance in science-related applications.

Signed Numbers

Each integer is said to be a signed number, which has either a positive sign or a negative sign.

Only the number 0 may be considered as being either positive or negative.

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Conventions where n is a natural number:

1. Any signed number n with no visible sign is assumed to be positive. $n = +n$

For example: $2 = +2$ $0 = +0$

Remember that 0 is an exception because it may be considered as either positive or negative depending on the context. Therefore: $0 = +0 = -0$

2.

The minus sign in front of a number n may be regarded as the product of (-1) times the number or: $-n = (-1) \cdot (n)$

For example: $-11 = (-1) \cdot (+11)$

or: $-11 = (-1) \cdot (11)$

3.

The plus sign in front of a number n may be regarded as the product of $(+1)$ times the number or: $+n = (+1) \cdot (n)$

For example: $+11 = (+1) \cdot (11)$

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The Absolute Value of a Signed Number

The absolute value of a signed number is the number itself without the sign.

By **convention 1**, it is always a positive number. Sometimes, it is regarded as the positive part of the signed number.

Notation:

$$| + n | = n = + n$$

$$| - n | = n = + n$$

$$| 0 | = 0 = + 0$$

where n is a natural number.

For example:

$$| + 5 | = 5 = + 5$$

$$| - 5 | = 5 = + 5$$

by convention 1

Note that the absolute value is **always** positive.

Multiplying Signed Numbers

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Multiplying Signed Numbers

What to do

The product of two signed numbers is found by multiplying the numbers (without the signs) together and then applying the Rule of Signs to the answer.

The Rule of Signs for Products

Any product of positive numbers is also a positive number.

The product of an **even** number of negative numbers is a positive number.

The product of an **odd** number of negative numbers is a negative number.

EXAMPLE 1

$$(4) \cdot (-1) = (+4) \cdot (-1) = -(4 \cdot 1) = -4$$

This is the justification for convention 2.

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Multiplying Signed Numbers

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EXAMPLE 2

$$(-7) \cdot (-2) \cdot (12) \cdot (-3) = - (7 \cdot 2 \cdot 12 \cdot 3) = -504$$

Each of these numbers is a factor of the entire product.

Three negative factors means a negative product.

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Multiplying Signed Numbers

What to do

The product of two signed numbers is found by multiplying the numbers (without the signs) together and then applying the Rule of Signs to the answer.

EXAMPLE 3

$$\begin{aligned} - (-8) \cdot (3) \cdot (-6) \cdot (-2) &= (-1) \cdot (-8) \cdot (3) \cdot (-6) \cdot (-2) \\ &= + (1 \cdot 8 \cdot 3 \cdot 6 \cdot 2) \\ &= + 288 \end{aligned}$$

Each of these numbers is called
a factor of the entire product.

Four negative factors
means a positive product.

EXAMPLE 4

$$\begin{aligned} - (- (- (- (-2)))) &= (-1) \cdot (-1) \cdot (-1) \cdot (-1) \cdot (-1) \cdot (2) \\ &= - (1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 2) = - 2 \end{aligned}$$

Five negative factors
means a negative product.

Exponentials of Signed Numbers

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The Exponential Operation of Signed Numbers What to do

The **exponential operation** is a **special case** of the operation of multiplication. An integer, or a signed number, when multiplied together with itself n times, is said to be taken to the exponent (or power) n .

The notation for the exponential operation is as follows:

$$\begin{array}{c} \text{exponent} \\ a^n := a \cdot a \cdot \dots \cdot a \\ \text{integer} \qquad \qquad \qquad n \text{ times} \end{array} \qquad \begin{array}{l} n \text{ natural number} \\ a \text{ signed number} \end{array}$$

All rules for multiplying signed numbers apply to evaluating exponentials of integers.

Multiplying Signed Numbers

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All rules for multiplying signed numbers apply to evaluating exponentials of integers.

EXAMPLE 1

What to do

$$\begin{array}{ccc} \text{exponent} & & \\ 4^4 & = & 4 \cdot 4 \cdot 4 \cdot 4 = +256 \\ \text{integer} & & 4 \text{ times} \end{array}$$

Any product of positive numbers is positive.

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The Exponential Operation of Signed Numbers

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All rules for multiplying signed numbers apply to evaluating exponentials of integers.

EXAMPLE 2

What to do

$$\begin{array}{c} \text{exponent} \\ (-4)^3 = (-4) \cdot (-4) \cdot (-4) = -64 \\ \text{integer} \qquad \qquad \qquad 3 \text{ times} \end{array}$$

An odd number of negative factors means a negative product.

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The Exponential Operation of Signed Numbers

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All rules for multiplying signed numbers apply to evaluating exponentials of integers.

EXAMPLE 3

What to do

$$\begin{array}{ccc} \text{exponent} & & \\ (-4)^4 & = & (-4) \cdot (-4) \cdot (-4) \cdot (-4) = +256 \\ \text{integer} & & 4 \text{ times} \end{array}$$

An even number of negative factors means a positive product.

Dividing Signed Numbers

What to do and what *not* to do

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Dividing Signed Numbers

What to do

At this initial stage of division, simply determine the sign of the numerator and the sign of the denominator separately.

Then apply the Rule of Signs for Quotients.

The Rule of Signs for Quotients

Any quotient of positive numbers is also a positive number.

The quotient of two negative numbers is a positive number.

The quotient of one negative number and one positive number is a negative number.

EXAMPLE 1

$$\left(\frac{11}{-6} \right) = - \frac{11}{6}$$

One negative sign in both numerator and denominator means a negative quotient.

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Dividing Signed Numbers

What to do

At this initial stage of division, simply determine the sign of the numerator and the sign of the denominator separately.

Then apply the Rule of Signs for Quotients.

The Rule of Signs for Quotients

Any quotient of positive numbers is also a positive number.

The quotient of two negative numbers is a positive number.

The quotient of one negative number and one positive number is a negative number.

EXAMPLE 2

$$\left(\frac{-7}{-4} \right) = + \frac{7}{4}$$

Two negative signs in both numerator and denominator means a positive quotient.

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Dividing Signed Numbers

At this initial stage of division, simply determine the sign of the numerator and the sign of the denominator separately.

Then apply the Rule of Signs for Quotients.

EXAMPLE 3

What *not* to do

Don't assume that a negative sign outside the fraction affects both numerator and denominator.

$$- \left(\frac{-7}{-4} \right) \neq \left(\frac{(-1) \cdot (-7)}{(-1) \cdot (-4)} \right) = + \frac{7}{4}$$

The negative sign may be applied to either numerator or denominator but NOT BOTH.

What to do

$$- \left(\frac{-7}{-4} \right) = \left(\frac{(-1) \cdot (-7)}{(-4)} \right) = \left(\frac{(-7)}{(-1) \cdot (-4)} \right) = - \frac{7}{4}$$

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Dividing Signed Numbers

What to do

At this initial stage of division, simply determine the sign of the numerator and the sign of the denominator separately.

Then apply the Rule of Signs for Quotients.

Alternate Rule of Signs for Quotients

If the number of negative factors in **both** numerator and denominator is **even**, then the quotient is positive.

If the number of negative factors in **both** numerator and denominator is **odd**, then the quotient is negative.

EXAMPLE 4

Three negative signs in both numerator and denominator means a negative quotient.

$$- \left(\frac{-7}{-4} \right) = (-1) \cdot \left(\frac{-7}{-4} \right) = -\frac{7}{4}$$

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Dividing Signed Numbers

What to do

At this initial stage of division, simply determine the sign of the numerator and the sign of the denominator separately.

Then apply the Rule of Signs for Quotients.

Alternate Rule of Signs for Quotients

If the number of negative factors in **both** numerator and denominator is **even**, then the quotient is positive.

If the number of negative factors in **both** numerator and denominator is **odd**, then the quotient is negative.

EXAMPLE 5

Four negative signs in both numerator and denominator means a positive quotient.

$$\frac{(-6) \cdot (-2) \cdot (9) \cdot (9)}{(7) \cdot (5) \cdot (-8) \cdot (-8)} = + \frac{972}{2240}$$

Adding Signed Numbers

What to do and what *not* to do

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Adding Signed Numbers

What to do

Case 1: when both numbers have the same sign

The sum of two signed numbers with the **same sign** is found by **adding** the numbers (without the signs) together and then applying the Rule of Signs to the answer.

Case 2: when the numbers have opposite signs

The sum of two signed numbers having **opposite signs** is found by **subtracting** the smaller number (without the sign) from the larger number (without the sign) and then applying the Rule of Signs to the answer.

The Rule of Signs for Sums

Any sum of positive numbers is also a positive number.

Any sum of negative numbers is also a negative number.

The sum of a positive number with a negative number will have the same sign as the larger (in absolute value) of the two numbers.

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Adding Signed Numbers

What to do

Case 1: when both numbers have the same sign

The sum of two signed numbers with the **same sign** is found by **adding** the numbers (without the signs) together and then applying the Rule of Signs to the answer.

The Rule of Signs for Sums

Any sum of positive numbers is also a positive number.

Any sum of negative numbers is also a negative number.

EXAMPLE 1

$$(4) + (17) = + (4 + 17) = + 21$$

$$(-9) + (-9) = - (9 + 9) = - 18$$

$$(-3) + (-1) + (0) + (-8) = - (3 + 1 + 0 + 8) = - 12$$

Each of these numbers is called
a term of the entire sum.

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Adding Signed Numbers

What to do

Case 2: when the numbers have opposite signs

The sum of two signed numbers having **opposite signs** is found by **subtracting** the smaller number (without the sign) from the larger number (without the sign) and then applying the Rule of Signs to the answer.

The Rule of Signs for Sums

The sum of a positive number with a negative number will have the same sign as the larger (in absolute value) of the two numbers.

EXAMPLE 2

$$(3) + (-7) = - (7 - 3) = -4$$

$$(-33) + (75) = + (75 - 33) = +42$$

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Adding Signed Numbers

Case 2: when the numbers have opposite signs

The sum of two signed numbers having **opposite signs** is found by **subtracting** the smaller number (without the sign) from the larger number (without the sign) and then applying the Rule of Signs to the answer.

The Rule of Signs for Sums

The sum of a positive number with a negative number will have the same sign as the larger (in absolute value) of the two numbers.

EXAMPLE 3 What to do

When there are more than two terms in a sum, add the numbers pairwise from left to right.

$$\begin{aligned} (-33) + (75) + (-43) &= + (75 - 33) + (-43) \\ &= + 42 + (-43) \\ &= - (43 - 42) = - 1 \end{aligned}$$

Subtracting Signed Numbers

What to do and what *not* to do

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Subtracting Signed Numbers

What to do

Convert the subtraction problem into an addition of signed numbers as follows:

$$(m) - (n) = (m) + (-n)$$

where m, n are signed numbers.

Then add the resulting signed numbers together according to the appropriate rules.

EXAMPLE 1

What to do

$$\begin{aligned} (-33) - (75) &= (-33) + (-75) \\ &= -(33 + 75) = -108 \end{aligned}$$

What not to do After bringing the minus sign into the bracket of the second term, do not forget to place a plus sign in between the two terms. Otherwise, the subtraction problem has incorrectly turned into a multiplication problem.

$$(-33) - (75) \neq (-33) (-75) = +2475$$

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Subtracting Signed Numbers

What to do

Convert the subtraction problem into an addition of signed numbers as follows:

$$(m) - (n) = (m) + (-n)$$

where m , n are signed numbers.

Then add the resulting signed numbers together according to the appropriate rules.

EXAMPLE 2

$$\begin{aligned}(3) - (-7) &= (3) + (-(-7)) = (3) + (+7) \\ &= + (3 + 7) = +10\end{aligned}$$

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Subtracting Signed Numbers

Convert the subtraction problem into an addition of signed numbers as follows:

$$(m) - (n) = (m) + (-n)$$

where m, n are signed numbers.

Then add the resulting signed numbers together according to the appropriate rules.

EXAMPLE 3 What to do

When there are two subtractions, one following the other, subtract the numbers pairwise from left to right.

$$\begin{aligned} (-33) - (75) - (-10) &= (-33) + (-75) - (-10) \\ &= (-108) - (-10) \\ &= (-108) + (+10) = -(108 - 10) \\ &= -98 \end{aligned}$$

